"Groundstates of nonlinear Choquard equations: Hardy–Littlewood–Sobolev critical exponent"

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Abstract

We consider nonlinear Choquard equation −Δu + Vu = (Iα * |u|^(α/N + 1))|u|^(α/N - 1) u where N ≥ 3, V ∈ L∞(ℝN) is an external potential and Iα(x) is the Riesz potential of order α ∈ (0, N). The power in the nonlocal part of the equation is critical with respect to the Hardy–Littlewood–Sobolev inequality. As a consequence, in the associated minimization problem a loss of compactness may occur. We prove that if liminf∞ (1-V(x))|x|²>N²(N - 2)/(4(N + 1)) then the equation has a nontrivial solution. We also discuss some necessary conditions for the existence of a solution. Our considerations are based on a concentration compactness argument and a nonlocal version of Brezis–Lieb lemma.

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GROUNDSTATES OF NONLINEAR CHOQUARD EQUATIONS: HARDY-LITTLEWOOD-SOBOLEV CRITICAL EXPONENT

VITALY MOROZ AND JEAN VAN SCHAFTINGEN

Abstract. We consider nonlinear Choquard equation

$$-\Delta u + Vu = \bigl(I_\alpha * |u|^{\frac{\alpha}{N}+1}\bigr)|u|^{\frac{\alpha}{N}-1}u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $V \in L^\infty(\mathbb{R}^N)$ is an external potential and $I_\alpha(x)$ is the Riesz potential of order $\alpha \in (0, N)$. The power $\frac{\alpha}{N}+1$ in the nonlocal part of the equation is critical with respect to the Hardy-Littlewood-Sobolev inequality. As a consequence, in the associated minimization problem a loss of compactness may occur. We prove that if $\liminf_{|x|\to\infty}\bigl(1-V(x)\bigr)|x|^2 > \frac{N^2(N-2)}{4(N+1)}$ then the equation has a nontrivial solution. We also discuss some necessary conditions for the existence of a solution. Our considerations are based on a concentration compactness argument and a nonlocal version of Brezis-Lieb lemma.

Contents

1. Introduction and results

We consider a nonlinear Choquard type equation

$$-\Delta u + Vu = \bigl(I_\alpha * |u|^p\bigr)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{$\mathcal{P}$}$$

where $N \in \mathbb{N}$, $\alpha \in (0, N)$, $p > 1$, $I_\alpha : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is the Riesz potential of order $\alpha \in (0, N)$ defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{N/2}\Gamma(\frac{\alpha}{2})\,|x|^{N-\alpha}},$$

and $V \in L^\infty(\mathbb{R}^N)$ is an external potential.

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For $N = 3$, $\alpha = 2$ and $p = 2$ equation $(\mathcal{P})$ is the *Choquard-Pekar equation* which goes back to the 1954's work by S. I. Pekar on quantum theory of a Polaron at rest [6, Section 2.1; 20] and to 1976's model of P. Choquard of an electron trapped in its own hole, in an approximation to Hartree-Fock theory of one-component plasma [8]. In the 1990's the same equation reemerged as a model of self-gravitating matter [7, 19] and is known in that context as the *Schrödinger-Newton equation*.

Mathematically, the existence and qualitative properties of solutions of Choquard equation $(\mathcal{P})$ have been studied for a few decades by variational methods, see [8; 11; 12, Chapter III; 14] for earlier and [2–5, 13, 16–18] for recent work on the problem and further references therein.

The following sharp characterisation of the existence and nonexistence of nontrivial solutions of $(\mathcal{P})$ in the case of constant potential V can be found in [16].

Theorem 1 (Ground states of $(\mathcal{P})$ with constant potential [16, theorems 1 and 2])**.** *Assume that $V \equiv 1$. Then $(\mathcal{P})$ has a nontrivial solution $u \in H^1(\mathbb{R}^N) \cap L^{\frac{2Np}{N-\alpha}}(\mathbb{R}^N)$ with $\nabla u \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap L^{\frac{2Np}{N-\alpha}}_{\mathrm{loc}}(\mathbb{R}^N)$ if and only if $p \in (\frac{\alpha}{N} + 1, \frac{N+\alpha}{(N-2)_+})$.*

If $p \in [\frac{\alpha}{N} + 1, \frac{N+\alpha}{(N-2)_+}]$ then $H^1(\mathbb{R}^N) \subset L^{\frac{2Np}{N-\alpha}}(\mathbb{R}^N)$ by the Sobolev inequality, and moreover, every H^1–solution of $(\mathcal{P})$ belongs to $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $p \ge 1$ by a regularity result in [17, proposition 3.1]. This implies that the Choquard equation $(\mathcal{P})$ with a positive constant potential has no H^1–solutions at the end-points of the above existence interval.

In this note we are interested in the existence and nonexistence of solutions to $(\mathcal{P})$ with nonconstant potential V at the *lower critical exponent* $p = \frac{\alpha}{N} + 1$, that is, we consider the problem

$$-\Delta u + V u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u \quad \text{in } \mathbb{R}^N. \tag{$\mathcal{P}_*$}$$

The exponent $\frac{\alpha}{N} + 1$ is critical with respect to the Hardy-Littlewood-Sobolev inequality, which we recall here in a form of minimization problem

$$c_\infty = \inf\Big\{\int_{\mathbb{R}^N} |u|^2 \mid u \in L^2(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1} = 1\Big\} > 0.$$

Theorem 2 (Optimal Hardy-Littlewood-Sobolev inequality [9, theorem 3.1; 10, theorem 4.3])**.** *The infimum c_∞ is achieved if and only if*

$$u(x) = C\left(\frac{\lambda}{\lambda^2 + |x-a|^2}\right)^{N/2}, \tag{1.1}$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $\lambda \in (0, \infty)$ are parameters.

The form of minimizers in theorem 2 suggests that a loss of compactness in $(\mathcal{P}_*)$ may occur by translations and dilations.

In order to characterise the existence of nontrivial solutions for the lower critical Choquard equation $(\mathcal{P}_*)$ we define the critical level

$$c_* = \inf\Big\{\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \mid u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1} = 1\Big\}.$$

It can be checked directly that if $u \in H^1(\mathbb{R}^N)$ achieves the infimum c_*, then a multiple of the minimizer u is a weak solution of Choquard equation $(\mathcal{P}_*)$.

Using a Brezis-Lieb type lemma for Riesz potentials [16, lemma 2.4] and a concentration compactness argument (lemma 10), we establish our main abstract result.

Theorem 3 (Existence of a minimizer)**.** *Assume that* $V \in L^\infty(\mathbb{R}^N)$ *and*

$$\liminf_{|x|\to\infty} V(x) \geq 1. \tag{1.2}$$

If $c_* < c_\infty$ *then the infimum* c_* *is achieved and every minimizing sequence for* c_* *up to a subsequence converges strongly in* $H^1(\mathbb{R}^N)$*.*

The inequality for the existence of minimizers is sharp, as shown by the following lemma for constant potentials.

Lemma 4. *If* $V \equiv 1$*, then* $c_* = c_\infty$*.*

Since problem $(\mathcal{P}_*)$ with $V \equiv 1$ has no H^1–solutions, this shows that the strict inequality $c_* < c_\infty$ is indeed essential for the existence of a minimizer for c_*.

In fact, the strict inequality $c_* < c_\infty$ is necessary at least for the strong convergence of *all* minimizing sequences.

Proposition 5. *Let* $V \in L^\infty(\mathbb{R}^N)$*. If*

$$\limsup_{|x|\to\infty} V(x) \leq 1,$$

then

$$c_* \leq c_\infty.$$

In addition, if

$$c_* = c_\infty,$$

then there exists a minimizing sequence for c_* *which converges weakly to* 0 *in* $H^1(\mathbb{R}^N)$*.*

Using Hardy-Littlewood-Sobolev minimizers (1.1) as a family of test functions for c_*, we establish a sufficient condition for the strict inequality $c_* < c_\infty$.

Theorem 6. *Let* $V \in L^\infty(\mathbb{R}^N)$*. If*

$$\liminf_{|x|\to\infty} \bigl(1 - V(x)\bigr)|x|^2 > \frac{N^2(N-2)_+}{4(N+1)}, \tag{1.3}$$

then $c_* < c_\infty$ *and hence the infimum* c_* *is achieved.*

In particular, if $N = 1, 2$ then condition (1.3) reduces to

$$\liminf_{|x|\to\infty} \bigl(1 - V(x)\bigr)|x|^2 > 0,$$

that is, the potential $1 - V$ should not decay to zero at infinity faster then the inverse square of $|x|$.

Employing a version of Pohožaev identity for Choquard equation $(\mathcal{P}_*)$ (see proposition 11 below), we show that a certain control on the potential V is indeed necessary for the strict inequality $c_* < c_\infty$.

Proposition 7. *Let $V \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If*

$$\sup\Big\{\int_{\mathbb{R}^N} \frac{1}{2}(\nabla V(x)|x)|\varphi(x)|^2 \,\mathrm{d}x \mid \varphi \in C^1_c(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |\nabla\varphi|^2 \le 1\Big\} < 1, \tag{1.4}$$

then Choquard equation $(\mathcal{P}_*)$ does not have a nonzero solution $u \in H^1(\mathbb{R}^N) \cap W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$.

In particular, combining (1.4) with Hardy's inequality on $\mathbb{R}^N$, we obtain a simple nonexistence criterion.

Proposition 8. *Let $N \ge 3$ and $V \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If for every $x \in \mathbb{R}^N$,*

$$\sup_{x \in \mathbb{R}^N} |x|^2(\nabla V(x)|x) < \frac{(N-2)^2}{2}, \tag{1.5}$$

then Choquard equation $(\mathcal{P}_*)$ does not have a nonzero solution $u \in H^1(\mathbb{R}^N) \cap W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$.

For example, for $N \ge 3$ and $\mu > 0$, we consider a model equation

$$-\Delta u + \Big(1 - \frac{\mu}{1 + |x|^2}\Big)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

Then proposition 8 implies that (1.6) has no nontrivial solutions for $\mu < \frac{(N-2)^2}{4}$, while for $\mu > \frac{N^2(N-2)}{4(N+1)}$ assumption (1.3) is satisfied and hence $(\mathcal{P}_*)$ admits a groundstate. We note that

$$\frac{\frac{(N-2)^2}{4}}{\frac{N^2(N-2)}{4(N+1)}} = 1 - \frac{N-2}{N^2},$$

so that the two bounds are asymptotically sharp when $N \to \infty$. We leave as an open question whether (1.6) admits a ground state for $\mu \in \big[\frac{(N-2)^2}{4}, \frac{N^2(N-2)}{4(N+1)}\big]$.

We emphasise that unlike the asymptotic sufficient existence condition (1.3), nonexistence condition (1.5) is a global condition on the whole of $\mathbb{R}^N$. For example, a direct computation shows that for $a = 0$ and every $\lambda > 0$, a multiple of the Hardy-Littlewood-Sobolev minimizer (1.1) solves the equation

$$-\Delta u + \Big(1 + \frac{N(2|x|^2 - N\lambda^2)}{(|x|^2 + \lambda^2)^2}\Big)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u \quad \text{in } \mathbb{R}^N. \tag{1.7}$$

Here (1.5) fails on an annulus centered at the origin, while $V(x) > 1$ and $(\nabla V(x)|x) < 0$ for all $|x|$ sufficiently large. Moreover,

$$\lim_{|x| \to \infty} (1 - V(x))|x|^2 = -2N < 0 \le \frac{N^2(N-2)_+}{4(N+1)}.$$

Note that the constructed solution u_λ satisfies

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + V|u_\lambda|^2 = 0.$$

In particular, we are unable to conclude that $c_* < c_\infty$. We do not know whether u_λ is a groundstate of (1.7). However, if u_λ was not a groundstate, then we would have $c_* < c_\infty$ and (1.7) would then have a groundstate by theorem 3.

2. Existence of minimizers under strict inequality: proof of theorem 3

In order to prove theorem 3 we will use a special case of the classical Brezis-Lieb lemma [1] for Riesz potentials.

Lemma 9 (Brezis-Lieb lemma for the Riesz potential [16, lemma 2.4])**.** *Let $N \in \mathbb{N}$, $\alpha \in (0, N)$, and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(\mathbb{R}^N)$. If $u_n \to u$ almost everywhere on $\mathbb{R}^N$ as $n \to \infty$, then*

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1} = \lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1} - \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^{\frac{\alpha}{N}+1})|u_n - u|^{\frac{\alpha}{N}+1}.$$

Our second result is a concentration type lemma.

Lemma 10. *Assume that $V \in L^\infty(\mathbb{R}^N)$ and $\liminf_{|x|\to\infty} V(x) \geq 1$. If the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^N)$ and converges in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ to u as $n \to \infty$, then*

$$\int_{\mathbb{R}^N} V|u|^2 \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V|u_n|^2 - \int_{\mathbb{R}^N} |u_n - u|^2.$$

Proof. Since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^N)$ and converges in measure to u, we deduce by the Brezis-Lieb lemma [1] (see also [10, theorem 1.9]) that

$$\int_{\mathbb{R}^N} V|u|^2 = \lim_{n\to\infty} \int_{\mathbb{R}^N} V|u_n|^2 - \int_{\mathbb{R}^N} V|u_n - u|^2.$$

Now, we observe that for every $R > 0$ and every $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} (1 - V)|u_n - u|^2 \leq \int_{B_R} (1 - V)|u_n - u|^2 + (1 - \inf_{\mathbb{R}^N\setminus B_R} V)_+ \int_{\mathbb{R}^N} |u_n - u|^2.$$

By the local $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ convergence, we note that

$$\lim_{n\to\infty} \int_{B_R} (1 - V)|u_n - u|^2 = 0.$$

Since $\lim_{R\to\infty}(1 - \inf_{\mathbb{R}^N\setminus B_R} v)_+ = 0$ and $(u_n - u)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^N)$, we conclude that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} (1 - V)|u_n - u|^2 \leq 0;$$

the conclusion follows. □

Proof of theorem 3. Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ be a minimizing sequence for c_*, that is

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1} = 1$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V|u_n|^2 \to c_*.$$

In view of our assumption (1.2) we observe that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. So, there exists $u \in H^1(\mathbb{R}^N)$ such that, up to a subsequence, the sequence $(u_n)_{n\in\mathbb{N}}$ converges to u weakly in $H^1(\mathbb{R}^N)$ and, by the classical Rellich-Kondrachov compactness theorem, strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. By the lower semi-continuity of the norm under weak convergence,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \le \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V|u_n|^2 = c_*.$$

and by Fatou's lemma

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1} \le 1.$$

In order to conclude, it suffices to prove that equality is achieved in the latter inequality.

We observe that by lemma 9,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^{\frac{\alpha}{N}+1})|u_n - u|^{\frac{\alpha}{N}+1} = 1 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}$$

while by lemma 10 and by the lower-semicontinuity of the norm under weak convergence,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 &\le \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \liminf_{n\to\infty} \int_{\mathbb{R}^N} V|u_n|^2 - |u_n - u|^2\\ &\le \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V|u_n|^2 - |u_n - u|^2\\ &= c_* - \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u|^2.\end{aligned} \tag{2.1}$$

By definition of c_∞, we have

$$\int_{\mathbb{R}^N} |u_n - u|^2 \ge c_\infty \Bigl(\int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^{\frac{\alpha}{N}+1})|u_n - u|^{\frac{\alpha}{N}+1}\Bigr)^{\frac{N}{N+\alpha}}.$$

Therefore, we conclude that

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \le c_* - c_\infty \Bigl(1 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}\Bigr)^{\frac{N}{N+\alpha}}.$$

In view of the definition of c_* this implies that

$$c_* \ge c_\infty \Bigl(1 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}\Bigr)^{\frac{N}{N+\alpha}} + c_* \Bigl(\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}\Bigr)^{\frac{N}{N+\alpha}}.$$

Since by assumption $c_* < c_\infty$, we conclude that

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1} = 1,$$

and hence, by definition of c_*,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 = c_*,$$

that is the infimum c_* is achieved at u. Moreover, from (2.1) we conclude that $u_n \to u$ in $L^2(\mathbb{R}^N)$. Since $V \in L^\infty(\mathbb{R}^N)$, this implies that $Vu_n \to Vu$ in $L^2(\mathbb{R}^N)$. Using (2.1) again, we conclude that

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2.$$

Since $(u_n)_{n\in\mathbb{N}}$ converges to u weakly in $H^1(\mathbb{R}^N)$, this implies that $(u_n)_{n\in\mathbb{N}}$ also converges to u strongly in $H^1(\mathbb{R}^N)$. □

3. Optimality of the strict inequality

In this section we prove lemma 4 and proposition 5.

Proof of lemma 4. Let us denote by $\tilde{c}_\infty$ the infimum on the right-hand side. By density of the space $H^1(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ and by continuity in L^2 of the integral functionals involved in the definition of c_∞, it is clear that $\tilde{c}_\infty \geq c_\infty$. We choose now $u \in H^1(\mathbb{R}^N)$ and define for $\lambda > 0$ the function $u_\lambda \in H^1(\mathbb{R}^N)$ for every $x \in \mathbb{R}^N$ by

$$u_\lambda(x) = \lambda^{N/2} u(\lambda x).$$

We compute for every $\lambda > 0$ that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^{\frac{\alpha}{N}+1}) |u_\lambda|^{\frac{\alpha}{N}+1} = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1}$$

and

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + |u_\lambda|^2 = \frac{1}{\lambda^2} \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^2.$$

Hence,

$$\inf_{\lambda>0} \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + |u_\lambda|^2 = \int_{\mathbb{R}^N} |u|^2,$$

and we conclude that $\tilde{c}_\infty \leq c_\infty$. □

Proof of proposition 5. For $\lambda > 0$, let

$$u_\lambda(x) = C \left(\frac{\lambda}{\lambda^2 + |x|^2} \right)^{\frac{N}{2}} = \lambda^{-\frac{N}{2}} u_1\Big(\frac{x}{\lambda}\Big)$$

be a family of minimizers for c_∞ given in (1.1). We observe that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^{\frac{\alpha}{N}+1}) |u_\lambda|^{\frac{\alpha}{N}+1} = 1,$$

whereas by a change of variables,

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + V|u_\lambda|^2 = \frac{1}{\lambda^2} \int_{\mathbb{R}^N} |\nabla u_1|^2 + \int_{\mathbb{R}^N} V\Big(\frac{y}{\lambda}\Big) \frac{C^2}{1+|y|^2} \,\mathrm{d}y.$$

By Lebesgue's dominated convergence theorem

$$\limsup_{\lambda \to 0} \int_{\mathbb{R}^N} V\Big(\frac{y}{\lambda}\Big) \frac{C^2}{1+|y|^2} \,\mathrm{d}y \leq \int_{\mathbb{R}^N} \frac{C^2}{1+|y|^2} \,\mathrm{d}y = c_\infty,$$

so we conclude that $c_* \leq c_\infty$. If, in addition, $c_* = c_\infty$ then for any $\lambda_n \to 0$, $(u_{\lambda_n})_{n \in \mathbb{N}}$ is a minimizing sequence for c_*, and the conclusion follows. □

4. Sufficient conditions for the strict inequality: proof of theorem 6

For $a \in \mathbb{R}^N$ and $\lambda > 0$, let

$$u_\lambda(x) = C \left(\frac{\lambda}{\lambda^2 + |x-a|^2} \right)^{N/2}$$

be a family of minimizers for c_∞ as in (1.1). Then

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + V|u_\lambda|^2 = c_\infty + \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + \int_{\mathbb{R}^N} (V-1)|u_\lambda|^2.$$

Denote

$$\mathcal{I}_V(a,\lambda) := \lambda^2 \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + \lambda^2 \int_{\mathbb{R}^N} (V-1)|u_\lambda|^2 < 0.$$

To obtain a sufficient conditions for $c_* < c_\infty$ it is enough to show that for some $a \in \mathbb{R}^N$,

$$\inf_{\lambda>0} \mathcal{I}_V(a,\lambda) < 0, \tag{4.1}$$

Proof of theorem 6. If $N \le 2$, then by (1.3) there exists $\mu > 0$ such that

$$\liminf_{|x|\to\infty} (1 - V(x))|x|^2 \ge \mu.$$

Therefore

$$\lim_{\lambda\to\infty} \lambda^2 \int_{\mathbb{R}^N} (1-V)|u_\lambda|^2 = \lim_{\lambda\to\infty} \int_{\mathbb{R}^N} \frac{\lambda^2(1-V(\lambda x))}{(1+|x|^2)^N} \,\mathrm{d}x \ge \int_{\mathbb{R}^N} \frac{\mu}{|x|^2(1+|x|^2)^N} \,\mathrm{d}x = \infty.$$

Since for every $\lambda > 0$,

$$\lambda^2 \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 = \int_{\mathbb{R}^N} |\nabla u_1|^2 < \infty,$$

the condition (4.1) is satisfied.

If $N \ge 3$, we observe that for every $\lambda > 0$,

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 = \frac{N^2(N-2)}{4(N+1)} \int_{\mathbb{R}^N} \frac{|u_\lambda(x)|^2}{|x|^2} \,\mathrm{d}x.$$

This follows from the fact that

$$\int_{\mathbb{R}^N} \frac{|x|^2}{(1+|x|^2)^{N+2}} \,\mathrm{d}x = \frac{N-2}{4(N+1)} \int_{\mathbb{R}^N} \frac{1}{|x|^2(1+|x|^2)^N} \,\mathrm{d}x,$$

which can be proved by two successive integrations by parts. Then, after a transformation $x = \lambda y + a$,

$$\mathcal{I}_V(a,\lambda) = \int_{\mathbb{R}^N} \Big(\frac{\frac{N^2(N-2)}{4(N+1)}}{|y|^2} - \lambda^2 (1 - V(a+\lambda y)) \Big) \frac{C^2}{(1+|y|^2)^N} \,\mathrm{d}y,$$

and in view of (1.3), sufficient condition is (4.1) is satisfied for $a = 0$, so we conclude that $c_* < c_\infty$. $\square$

Note that if the function $\lambda \mapsto \lambda^2(1 - V(a+\lambda y))$ is nondecreasing for every $y \in \mathbb{R}^N$, then $\lambda \mapsto \mathcal{I}_V(a,\lambda)$ is nonincreasing. Therefore $\mathcal{I}_V(a,\lambda)$ admits negative values if and only if it has a negative limit as $\lambda \to \infty$. The latter is ensured in theorem 6 via asymptotic condition (1.3). This explains that if the function $\lambda \mapsto \lambda^2(1 - V(a+\lambda y))$ is nondecreasing, like for instance, in the special case

$$V(x) = 1 - \frac{\mu}{1+|x|^2},$$

then integral sufficient condition (4.1) is in fact equivalent to the asymptotic sufficient condition (1.3).

5. Pohožaev identity and necessary conditions for the existence

We establish a Pohožaev type identity, which extends the identities (5.1) obtained previously for constant potentials V [4, lemma 2.1; 15; 16, proposition 3.1; 17, theorem 3].

Proposition 11. *Let $N \geq 3$ and $V \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $u \in W^{1,2}(\mathbb{R}^N)$. If*

$$\sup_{x \in \mathbb{R}^N} |(\nabla V(x)|)x| < \infty,$$

and $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies Choquard equation $(\mathcal{P}_*)$ then

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x)|x)\, |u(x)|^2 \,\mathrm{d}x.$$

Proof. We fix a cut-off function $\varphi \in C^1_c(\mathbb{R}^N)$ such that $\varphi = 1$ on B_1 and we test for $\lambda \in (0, \infty)$ the equation against the function $v_\lambda \in W^{1,2}(\mathbb{R}^N)$ defined for every $x \in \mathbb{R}^N$ by

$$v_\lambda(x) = \varphi(\lambda x)(\nabla u(x)|x)$$

to obtain the identity

$$\int_{\mathbb{R}^N} (\nabla u|\nabla v_\lambda) + \int_{\mathbb{R}^N} V u v_\lambda = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1} u v_\lambda.$$

We compute for every $\lambda > 0$, by definition of v_λ, the chain rule and by the Gauss integral formula,

$$\begin{aligned}
\int_{\mathbb{R}^N} V u v_\lambda &= \int_{\mathbb{R}^N} V(x) u(x) \varphi(\lambda x)(x|\nabla u(x)) \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} V(x)\varphi(\lambda x)(x|\nabla(\tfrac{|u|^2}{2})(x)) \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^N} \big((NV(x) + (\nabla V(x)|x))\varphi(\lambda x) + V(x)\lambda(x|\nabla\varphi(\lambda x))\big) \frac{|u(x)|^2}{2} \,\mathrm{d}x.
\end{aligned}$$

Since $\sup_{x\in\mathbb{R}^N} (\nabla V(x)|x) < \infty$, by Lebesgue's dominated convergence theorem it holds

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^N} V u v_\lambda = -\frac{N}{2} \int_{\mathbb{R}^N} V|u|^2 - \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x)|x)|u|^2.$$

By Lebesgue's dominated convergence again, since $u \in W^{1,2}(\mathbb{R}^N)$, we have (see [16, proof of proposition 3.1] for the details)

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^N} (\nabla u|\nabla v_\lambda) = -\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2.$$

and

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1} u\, v_\lambda = -\frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}.$$

We have thus proved the Pohožaev type identity

$$\begin{aligned}
\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2} \int_{\mathbb{R}^N} V|u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x)|x)\, |u(x)|^2 \,\mathrm{d}x \\
= \frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}.
\end{aligned} \tag{5.1}$$

If we test the equation against u, we obtain the identity
$$\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V|u|^2 = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1};$$
the combination of those two identities yields the conclusion. □

Proof of propositions 7 and 8. Proposition 7 is a direct consequence of proposition 11, while proposition 8 follows from proposition 11 and the classical optimal Hardy inequality on $\mathbb{R}^N$,
$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} \,\mathrm{d}x \le \int_{\mathbb{R}^N} |\nabla u|^2$$
which is valid for all $u \in H^1(\mathbb{R}^N)$ (see for example [21, theorem 6.4.10 and exercise 6.8]). □

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